Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Each of the four combinations of signs now gives rise to two solutions, which in regard to method do not differ.

If the distances to be cut off, instead of being equal, shall have given ratios

$$m : n : p$$

the method of solution remains the same.

The solution of this problem becomes more simple when two of the given lines are parallel.

SOLUTIONS OF PROBLEMS IN NUMBER FIVE.

SOLUTIONS of problems in No. 5 have been received as follows:

From Prof. L. G. Barbour, 224; Marcus Baker, 225, 230; Prof. P. E. Chase, 230; Newton Fitz, 221, 230; Henry Gunder, 221, 223, 224, 230; Henry Heaton, 221, 223, 224, 225, 226, 227, 228, 229, 230; W. E. Heal, 221, 227; Prof. E. W. Hyde, 225, 230; Prof. Joseph H. Kershner, 221, 222, 223, 224, 225, 226, 228, 229, 230; Chas. H. Kummell, 221, 224, 227, 228, 229, 230; Prof. D. J. Mc Adam, 223, 224, 228; Prof. Orson Pratt, 221; P. Richardson, 221, 225; Prof. J. Scheffer, 221, 224, 225, 226, 228; E. B. Seitz, 221, 223, 224, 225, 227, 228, 230.

---

221. “Find the four roots of the equation,

$$x^4+px^3+\frac{7p^2}{4^2}x^2+\frac{6p^3}{4^3}x+q=0.\text{”}$$

SOLUTION BY PROF. ORSON PRATT, SEN., SALT LAKE CITY, UTAH.

Increase the roots by $\frac{1}{4}p$, and the second and fourth terms vanish, and the resulting equation becomes a quadratic; reducing, and diminishing the roots by $\frac{1}{4}p$, we have

$$x=-\frac{p}{4}+\frac{1}{4\sqrt{2}}\sqrt{\left[-p^2+\sqrt{(9p^4-4^5q)}\right]};$$

$$x=-\frac{p}{4}-\frac{1}{4\sqrt{2}}\sqrt{\left[-p^2+\sqrt{(9p^4-4^5q)}\right]};$$

$$x=-\frac{p}{4}+\frac{1}{4\sqrt{2}}\sqrt{\left[-p^2-\sqrt{(9p^4-4^5q)}\right]};$$

$$x=-\frac{p}{4}-\frac{1}{4\sqrt{2}}\sqrt{\left[-p^2-\sqrt{(9p^4-4^5q)}\right]}.$$

SOLUTION BY P. RICHARDSON, BROOKLYN NEW YORK.

Put $p = 4t$, then the equation becomes

$$x^4+4tx^3+7t^2x^2+6t^3x+q = 0.$$

Put $x = y-t$, then is

$$y^4+t^2y^2-2t^4+q = 0;$$

$$\therefore \quad y = \pm \surd[-\tfrac{1}{2}t^2 \pm \surd(\tfrac{9}{4}t^4-q)].$$

But $y-t = x$;

$$\therefore \quad x = -t \pm \surd[-\tfrac{1}{2}t^2 \pm \surd(\tfrac{9}{4}t^4-q)].$$

---

222. "It is required to find four positive integral numbers, the sum of the cubes of any three of which shall be a rational cube number."

SOLUTION BY PROF. JOSEPH H. KERSHNER, MERCERSBERG, PA.

Let ax^2, bx^2, cx^2, dx^2 be the numbers. If $a+b+c = x$, $a+c+d = x$, $a+b+d = x$, and $b+c+d = x$, the conditions are fulfilled. Adding these equations, $x = \frac{3}{4}(a+b+c+d)$.

Assume $x-\frac{3}{4}a = p^3$, $x-\frac{3}{4}b = q^3$, $x-\frac{3}{4}c = r^3$, and $x-\frac{3}{4}d = s^3$, then

$$x = \tfrac{1}{3}(4x-x) = \tfrac{1}{3}(p^3+q^3+r^3+s^3).$$

Let $3m = p$, $3m+3 = q$, $3m+6 = r$, and $6m = s$; thus

$$\begin{aligned}
a &= \tfrac{4}{9}(-2p^3+q^3+r^3+s^3),\\
b &= \tfrac{4}{9}(+p^3-2q^3+r^3+s^3),\\
c &= \tfrac{4}{9}(+p^3+q^3-2r^3+s^3),\\
d &= \tfrac{4}{9}(+p^3+q^3+r^3-2s^3).
\end{aligned}$$

Therefore, if $p^3+q^3+s^3 > 2r^3$, or $(3m)^3+(3m+3)^3+(6m)^3 > (3m+6)^3$, a, b, c, d, and x shall all be positive and integral; that is,

$$8(3m)^3-27(3m)^2-189 \times 3m-405 > 0.$$

If $m = 3$, or more, the four numbers are known.

[The above is the only solution of problem 222 that has been received, hence we insert it, though we think it is defective.

In assuming $x = a+b+c = a+c+d$, &c., it is virtually assumed that $a = b = c = d$, consequently $p = q = r = s$; therefore the assumption, $3m = p$, $3m+3 = q$, &c., is not admissible —Ed.]

---

223. "Compare the masses of two spheres of equal radii, one of uniform density, the other increasing in density from a single point in the surface as the square of the distance measured on the diameter through that point increases."

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

Let m be the mass of the sphere of uniform density, m', the mass of the other sphere, r the radius of each sphere, and take a unit of volume of the sphere of uniform density for the unit of mass, and let the density of the second sphere, at the distance a from the point in the surface, be equal to the density of the first sphere.

Then $m = \frac{4}{3}\pi r^3$, and the mass of an elemental segment of the second sphere whose base is perpendicular to the diameter through the point in the surface, and is distant x from that point, is

$$\frac{\pi x^2}{a^2}\left(2rx - x^2\right)dx; \ \therefore\ m' = \int_0^{2r} \frac{\pi x^2}{a^2}\left(2rx - x^2\right)dx = \frac{8\pi r^5}{5a^2}.$$

$$\therefore\ m : m' :: 5a^2 : 6r^2.$$

---

224. "Two posts, each 12 ft. high, stand perpendicularly, fifty feet apart, on a horizontal plane, and a rope, one end of which is fastened at the top of each post, is suspended between them so that it just touches the plane at its middle point. Required the length of the rope."

SOLUTION BY HENRY GUNRER, NORTH MANCHESTER, IND.

The equations to the catenary are

$$y = -a + \frac{a}{2}\left(\varepsilon^{\frac{x}{a}} + \varepsilon^{-\frac{x}{a}}\right) \ldots (1), \text{ and } z = \frac{a}{2}\left(\varepsilon^{\frac{x}{a}} - \varepsilon^{-\frac{x}{a}}\right), \qquad (2)$$

in which x and y are the co-ordinates of the curve, the origin being at the lowest point, and the axis of x being tangent to the curve at the origin; and where a is the length of a portion of the robe the weight of which is equal the tension at the origin, and z, its length from the origin to the point (x, y).

Putting $x = 25$ and $y = 12$, (1) gives $a = 27.8$ ft., nearly, and putting this value of a in (2) we get $2z = 56.9$ ft., nearly, the length required.

SOLUTION BY E. B. SEITZ.

Let $h = 12$ feet, $2b = 50$ feet, $2l =$ the length of the rope, and let $a =$ the tension at the lowest point of the rope.

Then from the properties of the catenary we have

$$l^2 = h^2 + 2ah, \qquad (1)$$

$$b = a \log \left\{ \left[l + \sqrt{(l^2 + a^2)}\right] \div a \right\}. \qquad (2)$$

From (1) and (2), by eliminating a, we find

$$b=\left(\frac{l^2-h^2}{2h}\right)\log\left(\frac{l+h}{l-h}\right). \tag{3}$$

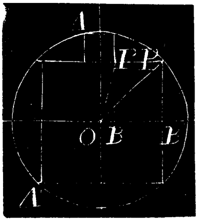
From (3), by the method of Double Position, we find

$$2l = 56.996 \text{ feet.}$$

---

225. "What is the locus of the angles of the inscribed squares of the respective segments of a circle when the chords or bases are parallel?"

BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.



$OB=x$, $PB=y$. Therefore for the point P we have

$$(2x+y)^2+x^2=R^2;$$

and for the point P', i. e., when x becomes half the side of the square in the greater segment, we have

$$(2x-y)^2+x^2=R^2;\ \therefore\ 5x^2\pm 4xy+y^2=R^2,$$

and consequently the locus is an *ellipse*.

---

226. "Find the algebraic equation whose roots are the real quantities found by giving integral values to k in

$$x=\cos\frac{2k\pi}{n},$$

n being a given integer. Also the equation whose solution is

$$x=\sin\frac{2k\pi}{n}."$$

SOLUTION BY PROF. J. SCHEFFER, MERCERSBERG, PA.

By De Moivre's formula we have

$$\left(\cos\frac{2k\pi}{n}+\sin\frac{2k\pi}{n}.\sqrt{-1}\right)^n=\cos 2k\pi+\sin 2k\pi\sqrt{-1}=1.$$

Developing the first member in a series by the Binomial Theorem, and putting $\cos(2k\pi\div n)=x$, we get the equation

$$x^n-\binom{n}{2}x^{n-2}\left(1-x^2\right)+\binom{n}{4}x^{n-4}\left(1-x^2\right)^2-\ldots-1=0;$$

of which in any concrete case we choose as many terms as is compatible with $n>r$ in $\binom{n}{r}$. Putting now $\sin\frac{2k\pi}{n}=x$, we get, if n is *odd*,

$$x^n-\binom{n}{2}x^{n-2}\left(1-x^2\right)+\binom{n}{4}x^{n-4}\left(x-{}^2\right)^2-\ldots \qquad =0;$$

and if n is *even*,

$$x^n-\binom{n}{2}x^{n-2}\left(1-x^2\right)+\binom{n}{4}x^{n-4}\left(1-x^2\right)^2-\ldots\pm 1=0;$$

where we use the upper sign if n is of the form $4m-2$, and the lower, if n is of the form $4m$.

SOLUTION BY H. HEATON, SABULA, IOWA.

Put $\frac{2k\pi}{n}=\theta$, and let $2x=2\cos\theta$. Then $2x=e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}$, and $2\sqrt{(x^2-1)}=2\sqrt{-1}.\sin\theta=e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}$; $\therefore\ e^{\theta\sqrt{-1}}=x+\sqrt{(x^2-1)}$ and $e^{-\theta\sqrt{-1}}=x-\sqrt{(x^2-1)}$. Also $2\cos n\theta=e^{n\theta\sqrt{-1}}+e^{-n\theta\sqrt{-1}}$

$$=[x+\sqrt{(x^2-1)}]^n+[x-\sqrt{(x^2-1)}]^n.$$

But $\cos n\theta=\cos 2k\pi=1$, k having any integral value. Hence the roots of the equation, $[x+\sqrt{(x^2-1)}]^n+[x-\sqrt{(x^2-1)}]^n=2$, are the different values of $\cos(2k\pi\div n)$.

If $x=\sin\theta$, $2\sqrt{-1}.x=e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}$, and $2\sqrt{(1-x^2)}=e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}$; $\therefore\ e^{\theta\sqrt{-1}}=\sqrt{-1}[x+\sqrt{(x^2-1)}]$ and $e^{-\theta\sqrt{-1}}=\sqrt{-1}[\sqrt{(x^2-1)}-x]$. Hence $2\sqrt{-1}.\sin n\theta=e^{n\theta\sqrt{-1}}-e^{-n\theta\sqrt{-1}}=(-1)^{\frac{1}{2}n}[x+\sqrt{(x^2-1)}]^n$

$$-(-1)^{\frac{1}{2}x}[\sqrt{(x^2-1)}-x]^n=0.$$

If n be even, and the expression $[x+\sqrt{(x^2-1)}]^n-[\sqrt{(x^2-1)}-x]^n$ be developed it will be found to be of the $2n$th degree, and its roots, when it equals nothing, are the different values of $2k\pi\div 2n$, k taking any integral value less than $2n$.

Therefore if n be even the required equation is $[\sqrt{(x^2-1)}+x]^{\frac{1}{2}n}-[\sqrt{(x^2-1)}-x]^{\frac{1}{2}n}=0$, or $[\sqrt{(x^2-1)}+x]^n+[\sqrt{(x^2-1)}-x]^n+2=0$. If n is odd the eq'n is $[\sqrt{(x^2-1)}+x]^n-[\sqrt{(x^2-1)}-x]^n=0$.

[Since the above was put in type we have receivd a solution of prob. 226 from Prof. Johnson, containing an interesting discussion of the required equations, which will appear, entire, in No. 1, Vol. VI.]

---

227. "If the equation $x^n+ax^{n-1}+\ldots=0$, whose roots are α, β, etc., be transformed into another, one of whose roots is 0, while the differences are unchanged, and then the transformed equation be divided by x; and if this process be repeated $n-1$ times, prove that the product of the differences of the roots of the original equation is equal to the product of the absolute terms of the equations thus obtained."

SOLUTION BY WILLIAM E. HEAL, WHEELING, INDIANA.

In the given equation, substitute $x—\alpha$ for x; then the roots of the equation will be $(\alpha—\alpha)$ $(=0)$, $(\alpha—\beta)$, $(\alpha—\gamma)$, &c. Hence the absolute term, being the product of these roots, will vanish and the equation will be divisible by x; and the absolute term of the new equation will be $(\alpha—\beta)(\alpha—\gamma)\times(\alpha—\delta)$ &c. For, from the manner of forming this term it must be equal to the sum of the terms formed by taking every combination of the quantities $(\alpha—\alpha)$, $(\alpha—\beta)$, $(\alpha—\gamma)$, $(\alpha—\delta)$, &c., taken in sets of $n—1$; but all the terms except that written above will contain the factor $(\alpha—\alpha)$ and will therefore be equal to zero.

Substituting $(x—\beta)$ for x in the given equation we find, in like manner, for the absolute term of the derived equation $(\beta—\alpha)(\beta—\gamma)(\beta—\delta)$ &c.

Hence, by repeating the substitution $n—1$ times and multiplying the absolute terms together, we obtain $(\alpha—\beta)(\beta—\alpha)(\alpha—\gamma)(\gamma—\alpha)(\alpha—\delta)(\delta—\alpha)$ &c., which is the product of differences of the given equation.

---

228. "Integrate $dI=\log(a+\sqrt{(x^2+b^2)}dx.$"

SOLUTION BY CHAS H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

Integrating by parts we have

$$I=x\log[a+\sqrt{(x^2+b^2)}]—\int\frac{x^2dx}{\sqrt{(x^2+b^2)}[a+\sqrt{(x^2+b^2)}]}$$

$$=x\log[a+\sqrt{(x^2+b^2)}]—I'. \qquad (1)$$

To rationalize I', assume

$$\sqrt{(x^2+b^2)}=y—x=\frac{y^2+b^2}{2y};\ x=\frac{y^2—b^2}{2y};\ dx=\frac{y^2+b^2}{2y^2}dy, \qquad (2)$$

then $$I'=\int\frac{(y^2—b^2)^2dy}{2y^2(y^2+2ay+b^2)}=\tfrac{1}{2}y—\tfrac{1}{2}\int dy\frac{2ay^3+3b^2y^2—b^4}{y^2(y^2+2ay+b^2)}$$

$$=\tfrac{1}{2}y—\tfrac{1}{2}I''. \qquad (3)$$

We have $$\frac{2ay^3+3b^2y^2—b^4}{y^2(y^2+2ay+b^2)}=—\frac{b^2}{y^2}+\frac{2a}{y}—\frac{4(a^2—b^2)}{y^2+2ay+b^2},$$

$$\therefore\quad I''=\frac{b^2}{y}+2a\log y—2\sqrt{(a^2—b^2)}\log\frac{y+a—\sqrt{(a^2—b^2)}}{y+a+\sqrt{(a^2—b^2)}};$$

$$\therefore\ \text{by (3)},\ I'=\tfrac{1}{2}y—\frac{b^2}{2y}—a\log y+\sqrt{(a^2-b^2)}\log\frac{y+a—\sqrt{(a^2—b^2)}}{y+a+\sqrt{(a^2—b^2)}},$$

$$= x - a\log[x+\sqrt{(x^2+b^2)}]+\sqrt{(a^2-b^2)}$$

$$\times \log\frac{x+\sqrt{(x^2+b^2)}+a-\sqrt{(a^2-b^2)}}{x+\sqrt{(x^2+b^2)}+a+\sqrt{(a^2-b^2)}} \text{ by (2).}$$

Therefore $I = x\log[a+\sqrt{(x^2+b^2)}]-x+a\log[x+\sqrt{(x^2+b^2)}]$

$$-\sqrt{(a^2-b^2)}\log\frac{a+x+\sqrt{(x^2+b^2)}-\sqrt{(a^2-b^2)}}{a+x+\sqrt{(x^2+b^2)}+\sqrt{(a^2-b^2)}},\ a>b.$$

---

229. "Evaluate

$$\int_{-\infty}^{\infty}\frac{\cos m\,x}{a+x}\,dx \text{ and } \int_{-\infty}^{\infty}\frac{\sin m\,x}{a+x}\,dx."$$

SOLUTION BY H. HEATON.

We have $\int_0^{\infty}\frac{\sin rx}{x}dx = \frac{\pi}{2}$. (See Todhunter's Integral Calculus, p. 271.)

Putting $-x$ for x, we have

$$\int_{-\infty}^{0}\frac{\sin rx}{x}dx = \frac{\pi}{2};\ \therefore \int_{-\infty}^{\infty}\frac{\sin rx}{x}dx = \pi.$$

Suppose $\int_0^{\infty}\frac{\cos rx}{x}dx = p$, and put $-x = x$; then is $\int_{-\infty}^{0}\frac{\cos rx}{x}dx = -p$.

$$\therefore \qquad \int_{-\infty}^{\infty}\frac{\cos rx}{x}dx = 0.$$

Put $y-a$ for x in the given problem; then

$$\int_{-\infty}^{\infty}\frac{\cos mx}{a+x}dx = \int_{-\infty}^{\infty}\frac{\cos ma\cos my+\sin ma\sin my}{y}dy = \pi\sin ma.$$

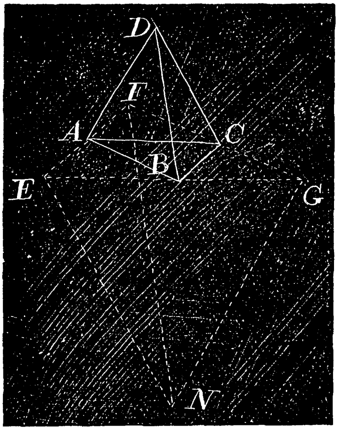
$$\int_{-\infty}^{\infty}\frac{\sin mx}{a+x}dx = \int_{-\infty}^{\infty}\frac{\cos ma\sin my-\sin ma\cos my}{y}dy = \pi\cos ma.$$

[Mr. Kummell has given an interesting discussion of these integrals which we are compelled to omit, at present, for want of room.]

---

230. "*ABCD* is any tetrahedron. Through *A* pass a plane parallel to *BCD*, through *B* a plane parallel to *CDA*, through *C* a plane parallel to *DAB* and through *D* a plane parallel to *ABC*. A new tetrahedron is thus formed the volume of which call V'. Also let V equal the volume of the original tetrahedron. Prove that $V' = 27\,V$."

SOLUTION BY NEWTON FITZ, NORFOLK, VIRGINIA.



Le ABC-D represent the tetrahedron V, and let a be its altitude.

A plane through A parallel to the plane DBC will cut the plane of the base in EF parallel to BC. A plane througn B parallel to DAC will cut the plane of the base in EG parallel to AC; and a plane through C parallel to DAB will cut the plane of the base in FG parallel to AB. The intersections of these three planes will be the lines EN, FN, GN, parallel respectively to DC, DB and DA.

The tetrahedron EFG-N will be similar to V and to V'. Because $AC = EB = BG$ the altitude of $EFG\text{-}N = 2a$. The altitude of the frustrum remaining to V' after EFG-N is removed is equal to the altitude of V, $= a$. Hence the altitude of $V' = 3a$ and $V' = 27V$. Q. E. D.

[Prof. Hyde's solution of this problem and also his solution of 225 are by quaternions. It was our intention to insert one of these solutions but the space remaining will not permit; and for like reason a solution of Mr. Baker's question, on p. 143, by Mr. Eastwood, and of Mr. Heal's question, on the same page, by Mr. Adcock, are, at present, excluded.]

PROBLEMMS.

231. By Prof. Orson Pratt, Sen. — What is the sum expressed in terms of m, of the values of all the determinants, from the second to the nth orders inclusive, which can be formed from the m-gonal series of numbers, represented by $1, a_2, a_3, a_4, \ldots a_n$, the arrangement of the constituants of the respective determinants being after the following form:

$$\begin{vmatrix} 1, & a_2 \\ a_3, & a_4 \end{vmatrix} + \begin{vmatrix} 1, & a_2, & a_3 \\ a_4, & a_5, & a_6 \\ a_7, & a_8, & a_9 \end{vmatrix} + \begin{vmatrix} 1, & a_2, & a_3, & a_4, \\ a_5, & a_6, & a_7, & a_8, \\ a_9, & a_{10}, & a_{11}, & a_{12}, \\ a_{13}, & a_{14}, & a_{15}, & a_{16}, \end{vmatrix} + \text{\&c.?}$$

232. By Prof. J. H. Kershner.—From two given points on a circle to draw straight lines through a point C in the circumference so they shall form with a line MN, given in position, a triangle CMN of given area.